

1. *Energy of a relativistic Fermi gas.* For electrons with energy $\varepsilon \gg mc^2$, where m is the rest mass of the electron, the energy is given by $\varepsilon \approx pc$ where p is the momentum. For electrons in a cube of volume $V = L^3$ the momentum is of the form $p = h(n_x^2 + n_y^2 + n_z^2)^{1/2}/2L$, which is the same as the nonrelativistic case.

(a) Show that in this extreme relativistic limit the Fermi energy of a gas of N electrons is given by

$$\varepsilon_F = \frac{hc}{2} \left(\frac{3N}{\pi V} \right)^{1/3}$$

(b) Show that the total energy of the ground state of the gas is

$$E_0 = \frac{3}{4} N \varepsilon_F$$

[Hint: Find the density of states for this relativistic gas and do the integral using the density of states.]

(a) We approximate the sum over states with an integral:

$$\sum_n (\dots) = \frac{1}{8} (2s+1) \int_0^\infty (\dots) 4\pi n^2 dn$$

where the factor of $1/8$ comes from only considering positive values of n_x , n_y , and n_z (the positive octant). The factor of $(2s+1)$, where s is the spin of the particles, comes from taking into account the additional spin states. Next, we need to express the n in terms of the energy. The energy for the extremely relativistic particles is

$$\varepsilon = pc = \frac{hcn}{2L} = \frac{hcn}{2V^{1/3}} \quad \rightarrow \quad n = \frac{2V^{1/3}}{hc} \varepsilon$$

The sum over states can then be approximated by

$$\sum_n (\dots) = \frac{\pi}{2} (2s+1) \int_0^\infty (\dots) \left(\frac{2V^{1/3}}{hc} \right)^3 \varepsilon^2 d\varepsilon = \frac{4\pi V}{(hc)^3} (2s+1) \int_0^\infty (\dots) \varepsilon^2 d\varepsilon$$

The density of states for this case (including the spin factor) is

$$D(\varepsilon) = \frac{4\pi V}{(hc)^3} (2s+1) \varepsilon^2$$

The Fermi energy is calculated by filling up energy levels until we have filled in N particles:

$$N = \int_0^{\varepsilon_F} D(\varepsilon) d\varepsilon = \frac{4\pi V}{(hc)^3} (2s+1) \int_0^{\varepsilon_F} \varepsilon^2 d\varepsilon = \frac{4\pi V}{(hc)^3} (2s+1) \frac{\varepsilon_F^3}{3}$$

Solving this for ε_F gives us

$$\varepsilon_F = \left[\frac{3(hc)^3 N}{4\pi V (2s+1)} \right]^{1/3} = \frac{hc}{2} \left[\frac{6N}{\pi V (2s+1)} \right]^{1/3} = \frac{hc}{2} \left(\frac{3N}{\pi V} \right)^{1/3}$$

where I plugged in $s = \frac{1}{2}$ in the last step.

(b) The energy in the ground state (for the N particle fermion gas) is

$$E_0 = \int_0^{\varepsilon_F} \varepsilon D(\varepsilon) d\varepsilon = \frac{8\pi V}{(hc)^3} \int_0^{\varepsilon_F} \varepsilon^3 d\varepsilon = \frac{8\pi V}{(hc)^3} \frac{\varepsilon_F^4}{4} = \left[\frac{8\pi V}{(hc)^3} \frac{\varepsilon_F^3}{3} \right] \frac{3}{4} \varepsilon_F = \frac{3}{4} N \varepsilon_F$$

2. Baierlein 9.4. A hypothetical system of N fermions has a single-particle density of states given by the linear relation $D(\varepsilon) = \varepsilon/\varepsilon_0^2$, where ε_0 is a positive constant with the dimensions of energy. The fermions do not interact among themselves. Calculate

- (a) the system's Fermi energy,
- (b) the chemical potential as a function of T under the conditions $0 \leq T \ll T_F$, and
- (c) the total energy $\langle E \rangle$ and heat capacity under the same assumptions.

In parts (b) and (c), it suffices to work through the lowest non-vanishing order in powers of the temperature, that is, through the first temperature-dependent term. Express all final answers in terms of N , ε_0 , k , and T .

(a) This is similar to the first problem. The Fermi energy is found from

$$N = \int_0^{\varepsilon_F} D(\varepsilon) d\varepsilon = \frac{1}{\varepsilon_0^2} \int_0^{\varepsilon_F} \varepsilon d\varepsilon = \frac{1}{\varepsilon_0^2} \frac{\varepsilon_F^2}{2} \quad \rightarrow \quad \varepsilon_F = \varepsilon_0 \sqrt{2N}$$

(b) Since the temperature is much less than the Fermi temperature, we can use the approximate formula, equation (9.15), for the chemical potential:

$$\mu = \varepsilon_F - \frac{\pi^2}{6} \frac{D'(\varepsilon_F)}{D(\varepsilon_F)} (kT)^2 = \varepsilon_F - \frac{\pi^2}{6} \frac{\frac{1}{2} \frac{\varepsilon_0}{\varepsilon_F^2}}{\frac{\varepsilon_F}{\varepsilon_0^2}} (kT)^2 = \varepsilon_F \left[1 - \frac{\pi^2}{6} \left(\frac{kT}{\varepsilon_F} \right)^2 \right] = \varepsilon_0 \sqrt{2N} \left[1 - \frac{\pi^2}{6} \frac{(kT)^2}{2N\varepsilon_0^2} \right]$$

(c) Again, since the temperature is much less than the Fermi temperature, we can use the approximate formula, equation (9.18), for the total energy:

$$\langle E \rangle = E_0 + \frac{\pi^2}{6} D(\varepsilon_F) (kT)^2 = E_0 + \frac{\pi^2}{6} \frac{\varepsilon_F}{\varepsilon_0^2} \frac{(kT)^2}{\varepsilon_F} = E_0 + \frac{\pi^2}{3} \left[\frac{\varepsilon_F^2}{2\varepsilon_0^2} \right] \frac{(kT)^2}{\varepsilon_F} = E_0 + \frac{\pi^2}{3} \frac{N (kT)^2}{\varepsilon_F}$$

The ground state energy is

$$E_0 = \int_0^{\varepsilon_F} \varepsilon D(\varepsilon) d\varepsilon = \frac{1}{\varepsilon_0^2} \frac{\varepsilon_F^3}{3} = \left[\frac{1}{\varepsilon_0^2} \frac{\varepsilon_F^2}{2} \right] \frac{2}{3} \varepsilon_F = \frac{2}{3} N \varepsilon_F$$

So we can write the energy as

$$\langle E \rangle = \frac{2}{3} N \varepsilon_F + \frac{\pi^2}{3} \frac{N (kT)^2}{\varepsilon_F} = \frac{2}{3} N \varepsilon_F \left[1 + \frac{\pi^2}{2} \left(\frac{T}{T_F} \right)^2 \right]$$

In terms of ε_0 and N , this is

$$\langle E \rangle = \frac{2^{3/2}}{3} N^{3/2} \varepsilon_0 \left[1 + \frac{\pi^2}{2} \frac{(kT)^2}{2N\varepsilon_0^2} \right] = \frac{2^{3/2}}{3} N^{3/2} \varepsilon_0 \left[1 + \frac{\pi^2}{4} \frac{(kT)^2}{N\varepsilon_0^2} \right]$$

The heat capacity is

$$C_V = \left(\frac{\partial \langle E \rangle}{\partial T} \right)_{V,N} = \frac{2^{1/2} \pi^2}{3} N^{1/2} k \frac{kT}{\varepsilon_0}$$

3. Baierlein 9.7. *Heating by adiabatic expansion.* Initially, an ideal gas of N fermions (of spin $\frac{1}{2}\hbar$) is confined to a volume V_i and has zero temperature. Then a valve is opened, and the gas expands adiabatically into an evacuated region; the combined spaces have a volume V_f . When the gas settles to a new thermal equilibrium, its behavior is found to be that of a classical ideal gas. (Note: The walls are rigid and insulating throughout.)
- What is the final temperature of the gas?
 - Derive an inequality for the ratio V_f/V_i and explain your reasoning.
 - The title of this problem is surely an oxymoron. Explain the sense in which the title is a contradiction in terms and also the sense in which the title is a legitimate use of words.

(a) Since this is done adiabatically and the gas does no work expanding into a vacuum, the total energy will be conserved. In the final classical ideal gas state, $E = \frac{3}{2}NkT$. In the zero temperature Fermi gas state the energy is $E = \frac{3}{5}N\varepsilon_F$ where ε_F is the Fermi energy. Setting these equal:

$$\frac{3}{2}NkT = \frac{3}{5}N\varepsilon_F \quad \rightarrow \quad T = \frac{2\varepsilon_F}{5k} = \frac{2}{5} \frac{h^2}{2mk} \left(\frac{3N}{8\pi V_i} \right)^{2/3} = \frac{h^2}{5mk} \left(\frac{3N}{8\pi V_i} \right)^{2/3}$$

(b) The new density must be such that the number of particles in a volume the size of the thermal deBroglie wavelength is small:

$$\begin{aligned} \frac{N}{V_f} \lambda_{\text{th}}^3 &\ll 1 \\ \frac{N}{V_f} \left(\frac{h}{\sqrt{2\pi mkT}} \right)^3 &\ll 1 \\ \frac{N}{V_f} \frac{h^3}{(2\pi m)^{3/2}} \left[\frac{5m}{h^2} \left(\frac{8\pi V_i}{3N} \right)^{2/3} \right]^{3/2} &\ll 1 \\ \frac{V_i}{V_f} \frac{8\pi}{3} \left(\frac{5}{2\pi} \right)^{3/2} &\ll 1 \quad \rightarrow \quad V_f \gg 6V_i \end{aligned}$$

where I have rounded off the numerical factor to one significant figure.

(c) No thermal energy was added to the gas in this process so it does not make sense to call this heating. On the other hand the temperature of the gas increased in the process and it did no work. Strictly speaking, this is *not* heating which implies an exchange of thermal energy.

4. Baierlein 9.14. *Stars with degenerate but non-relativistic electrons throughout.* Determine the proportionality constant in our theoretical relation, $R/R_\odot = \text{const} \times (M/M_\odot)^{-1/3}$, first algebraically and then numerically. How does the relationship fare when compared with the radius and mass of the white dwarf star 40 Eri B? (For a sense of scale, note that observed stellar masses span a wide range: from approximately $0.1M_\odot$, the least mass that can initiate fusion, to approximately $30M_\odot$, for extremely bright but short-lived stars.)

We have two equations for the pressure in the star:

$$P = \frac{2}{5}\varepsilon_F \frac{N}{V} = \frac{2}{5} \frac{h^2}{2m} \left(\frac{3}{8\pi} \right)^{2/3} \left(\frac{N}{V} \right)^{5/3} \quad P = \frac{1}{2} \frac{GM\rho}{R}$$

where the first equation assumes a uniform pressure and the second equation refers to the pressure at the center of the star. Also, $\rho = M/V$ is the average density. To make a rough estimate of the relation, we set these equal to each other.

$$\frac{h^2}{5m} \left(\frac{3}{8\pi} \right)^{2/3} \left(\frac{N}{V} \right)^{5/3} = \frac{1}{2} \frac{GM^2}{RV}$$

The number of electrons will be related to the total mass, assuming 2 nucleons per electron, $N \approx M/2m_p$ (one of the nucleons is a proton and one a neutron) and the volume is $V = \frac{4}{3}\pi R^3$.

$$\frac{h^2}{5m} \left(\frac{3}{8\pi}\right)^{2/3} \left[\frac{M}{2m_p \left(\frac{4}{3}\pi R^3\right)}\right]^{5/3} = \frac{1}{2} \frac{GM^2}{R \left(\frac{4}{3}\pi R^3\right)}$$

$$\frac{h^2}{5m} \left(\frac{3}{8\pi}\right)^{2/3} \left(\frac{3}{8\pi m_p}\right)^{5/3} \frac{M^{5/3}}{R^5} = \frac{3G}{8\pi} \frac{M^2}{R^4}$$

Solving for the radius:

$$R = \frac{h^2}{80mGm_p^{5/3}} \left(\frac{3}{\pi}\right)^{4/3} \frac{1}{M^{1/3}}$$

The radius clearly scales inversely with the mass to the one-third power. To put this in terms of the radius and mass relative to the solar radius and mass:

$$\frac{R}{R_\odot} = \frac{h^2}{80mGm_p^{5/3}} \left(\frac{3}{\pi}\right)^{4/3} \frac{1}{R_\odot M_\odot^{1/3}} \left(\frac{M_\odot}{M}\right)^{1/3} = \frac{3h^2}{80\pi mGm_p R_\odot} \left(\frac{3}{\pi m_p^2 M_\odot}\right)^{1/3} \left(\frac{M_\odot}{M}\right)^{1/3} = 0.0041 \left(\frac{M_\odot}{M}\right)^{1/3}$$

The mass of 40 Eri B is $0.447M_\odot$ which, using the above formula, would correspond to a radius of $0.0054R_\odot$. The actual radius of 40 Eri B is $0.013R_\odot$, about a factor of 2 higher. So our simple analysis is fairly close by astrophysical standards.

5. Baierlein 9.20. *BEC with sodium*. The second experiment to produce BEC in a dilute gas used sodium atoms. The number density was $N/V = 10^{20}$ atoms/m³. The mass of a sodium atom is $m = 3.82 \times 10^{-26}$ kg. As with the rubidium experiment, only one state of intrinsic angular momentum was populated.
- If the trap that confined the atoms were adequately approximated by a box with rigid walls, at what temperature would you expect BEC to set in (as one lowered the temperature)?
 - How low a temperature would be required for 90 percent of the atoms to be in the single-particle ground state?
 - The common, stable isotope of sodium has 12 neutrons and is the isotope referred to above: ²³Na. The unstable isotope ²¹Na has ten neutrons, the same nuclear spin, and a half-life of 23 seconds. In the following suppress the possibility radioactive decay.

A box of volume 1 cm³ contains 10^{14} sodium atoms at a temperature $T = 1.3 \times 10^{-6}$ K. The atoms form a dilute gas, and only one state of intrinsic angular momentum is populated. Determine whether the heat capacity C_V is an increasing or decreasing function of temperature if

- all atoms are ²³Na atoms;
- half are ²³Na and half are ²¹Na.

(a) The Bose-Einstein condensate temperature:

$$T_B = \frac{1}{\pi (2.612)^{2/3}} \frac{h^2}{2mk} \left(\frac{N}{V}\right)^{2/3} = 0.0839 \frac{(6.626 \times 10^{-34} \text{ J} \cdot \text{s})^2}{(3.82 \times 10^{-26} \text{ kg})(1.38 \times 10^{-23} \text{ J/K})} (10^{20} \text{ m}^{-3})^{2/3} = 1.5 \times 10^{-6} \text{ K}$$

(b) For $N_0/N = 0.9$, we must have:

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_B}\right)^{3/2} = 0.9 \quad \rightarrow \quad T = T_B (.1)^{2/3} = 3.2 \times 10^{-7} \text{ K}$$

(c) These are the same conditions as in part (a). For all ²³Na, this temperature is lower than the Bose-Einstein condensation temperature so the gas is in a condensate phase and the heat capacity is an *increasing* function of temperature. For a 50-50 split, the density of each type will be half of what it was in part (a).

Since T_B scales as the number density to the 2/3 power, this is a factor of $(0.5)^{2/3} = 0.63$. There will also be a factor of 23/21 for the ^{21}Na atoms due to the mass difference.

$$\begin{aligned} \text{For the } ^{23}\text{Na atoms: } T_B &= 0.945 \times 10^{-6} \text{ K} \\ \text{For the } ^{21}\text{Na atoms: } T_B &= 1.035 \times 10^{-6} \text{ K} \\ \text{Actual temperature: } T &= 1.3 \times 10^{-6} \text{ K} \end{aligned}$$

Since the two species are distinguishable (due to the mass) we must treat them as separate gases. In this case, neither would be in a condensate phase.

6. Baierlein 9.21. *Some details for the Bose-Einstein condensation.*

- Chemical potential when $T \leq 0.99T_B$. Solve equation (9.52) for the chemical potential μ when $T < T_B$. Show that the size of μ is consistent with our treatment of the integral in (9.47) provided $T \leq 0.99T_B$, to give a conservative upper limit. This calculation provides a self-consistency check.
- $\langle n_2 \rangle$ relative to $\langle n_1 \rangle$. Estimate $\langle n_2 \rangle$, in order of magnitude, when $T \leq 0.99T_B$. For one route, show first that $\varepsilon_2 - \mu = \varepsilon_1 + (\varepsilon_1 - \mu)$, that $\varepsilon_1 = kT_B/O(N^{2/3})$, and that $\varepsilon_1 - \mu = kT/O(N)$. The symbol “ $O(\dots)$ ” denotes “a number of order (...)”.
- $\langle n_1 \rangle$ when $T = T_B$. Computing $\langle n_1 \rangle$ when $T = T_B$ is a delicate business. Start by introducing shifted energies: $\varepsilon'_\alpha = \varepsilon_\alpha - \varepsilon_1$ and $\mu' = \mu - \varepsilon_1$. On the shifted scale, the energy of the lowest single particle state is zero, but the estimated occupation number remains as before. Equation (9.47) becomes

$$\frac{1}{e^{-\mu'/kT} - 1} + C(kT)^{3/2} \int_0^\infty \frac{x^{1/2}}{e^{x-\mu'/kT} - 1} dx = N.$$

The integral is $(\pi^{1/2}/2) 2.612$ when $-\mu'/kT = 0$. Its value *decreases* when $-\mu'/kT > 0$ because the divisor is larger. The integral has an expansion as

$$\frac{\pi^{1/2}}{2} \left[2.612 - 3.54 \left(-\frac{\mu'}{kT} \right)^{1/2} + \dots \right],$$

as derived by John E. Robinson, *Phys. Rev.* **83**, 678-9 (1951). Use this information to solve for $-\mu'/kT$ and then $\langle n_1 \rangle$, in order of magnitude, when $T = T_B$.

- Equation (9.52) gives us the expected occupation of the ground state as a function of temperature.

$$\langle n_1 \rangle = N \left[1 - \left(\frac{T}{T_B} \right)^{3/2} \right] = \frac{1}{e^{(\varepsilon_1 - \mu)/kT} - 1}$$

Solving this for μ :

$$\begin{aligned} e^{(\varepsilon_1 - \mu)/kT} - 1 &= \frac{1}{N \left[1 - \left(\frac{T}{T_B} \right)^{3/2} \right]} \quad \rightarrow \quad \frac{\varepsilon_1 - \mu}{kT} = \ln \left\{ 1 + \frac{1}{N \left[1 - \left(\frac{T}{T_B} \right)^{3/2} \right]} \right\} \\ \mu &= \varepsilon_1 - kT \ln \left\{ 1 + \frac{1}{N \left[1 - \left(\frac{T}{T_B} \right)^{3/2} \right]} \right\} \end{aligned}$$

When the natural log is of $1 +$ a small number, it is approximately equal to the small number.

$$\mu \approx \varepsilon_1 - \frac{kT}{N \left[1 - \left(\frac{T}{T_B} \right)^{3/2} \right]}$$

The second term must be very small in order for us to ignore the μ term in the integral. If $T = 0.99T_B$, we have

$$\mu \approx \varepsilon_1 - \frac{0.99kT_B}{N \left[1 - (0.99)^{3/2}\right]} = \varepsilon_1 - \frac{66kT_B}{N}$$

As long as N is large, this is still very small even at $T = 0.99T_B$.

(b) Recall that the energy levels as a function of n_x , n_y and n_z are

$$\varepsilon_n = \frac{h^2}{8mV^{2/3}} (n_x^2 + n_y^2 + n_z^2)$$

The ground state is $n_x = n_y = n_z = 1$ and the first excited state is with one of these equal to 2. The energies are

$$\varepsilon_1 = \frac{3h^2}{8mV^{2/3}} \quad \varepsilon_2 = \frac{6h^2}{8mV^{2/3}} = 2\varepsilon_1$$

In terms of the Bose condensation temperature,

$$kT_B = \frac{1}{\pi (2.612)^{3/2}} \frac{h^2}{2m} \left(\frac{N}{V}\right)^{2/3} = \frac{4N^{2/3}}{\pi (2.612)^{3/2}} \frac{h^2}{8mV^{2/3}},$$

the ground state energy is

$$\varepsilon_1 = \frac{3\pi (2.612)^{3/2} kT_B}{4N^{2/3}} = \frac{kT_B}{O(N^{2/3})}$$

We showed in the previous problem that

$$\varepsilon_1 - \mu = \frac{66kT_B}{N} = \frac{kT_B}{O(N)}$$

We can then estimate

$$\frac{\varepsilon_2 - \mu}{kT} = \frac{T_B}{T} \frac{\varepsilon_1 + (\varepsilon_1 - \mu)}{kT_B} = \frac{T_B}{T} \left[\frac{1}{O(N^{2/3})} + \frac{1}{O(N)} \right]$$

So for T near T_B

$$\langle n_2 \rangle = \frac{1}{e^{(\varepsilon_2 - \mu)/kT} - 1} = \frac{1}{e^{\varepsilon_1} e^{(\varepsilon_1 - \mu)/kT} - 1} \approx \frac{1}{\left(1 + \frac{T_B}{T} \frac{1}{O(N^{2/3})}\right) \left(1 + \frac{T_B}{T} \frac{1}{O(N)}\right) - 1} \approx \frac{T_B}{T} O(N^{2/3})$$

This is quite large near $T = T_B$ (but still much smaller than $\langle n_1 \rangle$). At lower temperatures, the factor T_B/T can become quite large and the number of particles in the first excited state becomes small (goes to zero as $T \rightarrow 0$).